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SHEAR WAVES IN HARDENING RIGIDLY PLASTIC BODIES*

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The dynamic flow of a hardening rigidly plastic medium with a flow law related to the Mises or Tresca yield condition is considered. The Odqvist parameter is taken as the hardening parameter (the Odqvist parameter is related to the specific plastic work w in the Mises theory by the dependence $dw = kd\chi$, where k is the shear yield limit). It is shown that in an arbitrary continuous medium the surface of weak velocity discontinuity (S) at each point should be tangent to the principal direction of the strain rate tensor, if the principal directions of this tensor are continuous. The flow law imposes new constraints on S. Thus, weak velocity discontinuities can only be on the maximal shear surface in the Mises theory.

As an illustration, the problem of plastic deformation propagation in a half-plane is considered when the velocity is given on an edge, where the solution shows that weak discontinuities can be caused by the boundary conditions. It turns out also that a continuous solution or one containing only weak velocity discontinuities is not always possible. In this connection, the problem of the structure of a strong velocity discontinuity (shock) is solved with viscosity taken into account. Existence conditions for the shock, and an equation governing its propagation velocity are obtained. The results obtained are applied to the problem of strain propagation in rigidly plastic bodies.

Among the large quantity of papers on the wave theory of plastic media, we note /1— 3/ as being closest to the theme of the present paper.

1. Kinematic conditions on surfaces of weak velocity discontinuity (S).

We show that independently of its properties, a surface S in a continuous medium should be tangent to one of the principal directions of the tensor ε_{ij} if the velocity field (v_i) and the principal directions of ε_{ij} are continuous on S.

Let D_1 and D_2 denote domains belonging to S. Let the velocity field v_i and the tensor ε_{ij} be known in some neighborhood of S at the time under consideration. We consider an arbitrary point M in S, at which the principal directions of the tensor ε_{ij} are uniquely defined. In the neighborhood of the point M in D_1 and D_2 we introduce a coordinate system η_i whose coordinate lines agree with the trajectories of the principal strain rates.

We write the strain rate tensor components in the form (H_i are Lamé parameters)

$$\varepsilon_{11} = \frac{\partial v_1}{\partial s_1} + v_2 \frac{\partial \lambda_1}{\partial s_2} + v_3 \frac{\partial \lambda_1}{\partial s_3}, \ldots, \quad 2\varepsilon_{12} = \gamma_{12} = \frac{\partial v_1}{\partial s_2} + \frac{\partial v_2}{\partial s_1} - v_1 \frac{\partial \lambda_1}{\partial s_2} - v_2 \frac{\partial \lambda_2}{\partial s_1} \tag{1.1}$$

$$(\lambda_i = \ln H_i, \partial / \partial s_i = H_i^{-1} \partial / \partial \eta_i)$$

The relationship $\gamma_{ij} = 0$ is satisfied on both sides of S in the coordinate system η_i. The coordinate system η_i can always be selected so that $v_i = 0$ at the point M. Then the relationships

$$\left[\frac{\partial v_2}{\partial s_1}\right] + \left[\frac{\partial v_1}{\partial s_2}\right] = 0, \quad \left[\frac{\partial v_3}{\partial s_2}\right] + \left[\frac{\partial v_2}{\partial s_3}\right] = 0, \quad \left[\frac{\partial v_1}{\partial s_3}\right] + \left[\frac{\partial v_3}{\partial s_1}\right] = 0 \tag{1.2}$$

should be satisfied at the point M. Here $[\partial v_i / \partial s_j]$ is the jump in $\partial v_i / \partial s_j$ on S. Setting $\partial v_i / \partial s_j = \mu_i n_j$, where $\mu_i = \partial v_i / \partial n$ is the jump in the derivative along the normal to the surface of discontinuity, and n_j are direction cosines of the normal, we obtain a system of equations in μ_i

$$\mu_1 n_2 + \mu_2 n_1 = 0, \quad \mu_2 n_3 + \mu_3 n_2 = 0, \quad \mu_1 n_3 + \mu_3 n_1 = 0 \tag{1.3}$$

There will be nonzero solutions under the condition $n_1 n_2 n_3 = 0$. Therefore, one of the principal directions of ε_{ij} is tangent to S.

2. Discontinuities in Mises theory.

Henceforth, only surfaces of weak velocity discontinuity S or, equivalently, the surface of strain rate tensor discontinuity, on which the principal directions of the stress tensor are continuous, and therefore, the principal directions of the tensor ε_{ij} are continuous. Such a constraint is justified by the conception of surfaces of stress discontinuity as degenerate rigid zones taken in quasistatics in the theory of plasticity, whereupon the tensor ε_{ij} cannot undergo a discontinuity thereon.

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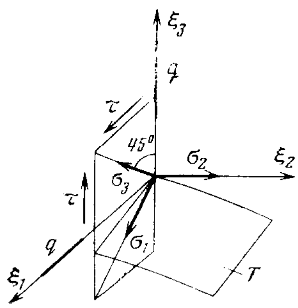

Fig.1

The flow law can impose additional constraints on the surface S.

Let σ_i be the principal stress, where $\sigma_1 \geqslant \sigma_2 \geqslant \sigma_3$. We introduce the curvilinear coordinates ξ_i. The direction of ξ_2 at each point is compatible with the direction of σ_2, and the directions ξ_1, ξ_3 are compatible with the directions of the areas of the extremal tangential stress (Fig.1). If $\sigma_1 = \sigma_2$, then any direction orthogonal to σ_3 can be taken as ξ_2. We proceed analogously in the case $\sigma_2 = \sigma_3$. The coordinate system ξ_i can be introduced in the neighborhood of any nonsingular point of the stress tensor.

Let S_{ij} be the stress tensor-deviator. We have $s_{11} = s_{33} = -{}^1/_2\, s_{22} = (q - \sigma_2) / 3$, $s_{12} = s_{23} = 0$, $s_{13} = \tau$.

The flow law yields (k is the shear yield point)

$$\varepsilon_{ij} = (\chi' / 2k)\, s_{ij} \tag{2.1}$$

Therefore, $\varepsilon_{11} = \varepsilon_{33} = -{}^1/_2\, \varepsilon_{22}$, $\gamma_{12} = \gamma_{23} = 0$. From this and from (1.1) we obtain the relationship on the surface of the weak velocity discontinuity S. (We shall now understand $\partial / \partial s_i$ to be the differentiation operator in the direction ξ_i). Reasoning exactly as in Sect.1, we arrive at the deduction that the surface S should be tangent at each of its points to one of the extremal shear areas. We call such surfaces shear surfaces.

Let us assume that the plasticity condition is satisfied on both sides of the surface of discontinuity. It is clear from the above that discontinuities of ε_{ij} are possible only on shear surfaces. Let us superpose the surface $\xi_3 = \mathrm{const}$ on the surface of discontinuity while temporarily not assuming that $\sigma_1 \geqslant \sigma_2 \geqslant \sigma_3$. The plasticity condition has the form $(q - \sigma_2)^2 + 3\tau^2 = 3k^2$. Since $[\tau] = [q] = 0$, then $[\sigma_2] = 0$ also. Since $[\varepsilon_{11}] = [\varepsilon_{22}] = 0$, then $\sigma_2 = q = {}^1/_2\, (\sigma_1 + \sigma_3)$ from (2.1). Therefore $[\varepsilon_{33}] = 0$.

Therefore, if $\xi_3 = \mathrm{const}$ is a surface of discontinuity of ε_{ij}, then $\sigma_1 > \sigma_2 > \sigma_3$. Therefore, the surface of discontinuity of ε_{ij} can only be a maximal shear surface.

Now, let $\sigma_1 > \sigma_2 > \sigma_3$. Then $\tau = k$, $\gamma_{13} = \gamma$ (γ is the maximal shear velocity). From (2.1) we obtain $[\gamma] = [\chi^{\cdot}]$.

The hardening condition in the coordinates ξ_i takes the form

$$\chi^{\cdot} = |\gamma| \tag{2.2}$$

Hence, the condition $[\gamma] = [\chi']$ also follows, which we rewrite in the form (G is the normal velocity of the surface of discontinuity)

$$(v_3 - G)\, [\partial \chi / \partial s_3] - [\gamma] = 0 \tag{2.3}$$

3. Dynamic compatibility conditions.

Let us evaluate the acceleration jump on the surface of weak velocity discontinuity S by superposing it on the surface $\xi_3 = \mathrm{const}$. We consider the coordinate system ξ_i as proper for a certain point M of the surface S. The acceleration of the point M will be $\mathbf{a} = d\mathbf{v} / dt = \partial \mathbf{v} / \partial t$. Hence $[a_i] = -G\, [\partial v_i / \partial s_3]$ (a_i is the projection of the acceleration in the direction ξ_i). It follows from (1.1) that for an arbitrary point of the surface S

$$[a_1] = (v_3 - G)\, [\gamma_{13}], \quad [a_2] = (v_3 - G)\, [\gamma_{23}], \quad [a_3] = [v_3 - G)\, [\varepsilon_{33}] \tag{3.1}$$

As has been shown earlier, in Mises theory $[\gamma_{13}] = [\gamma]$, $[\gamma_{23}] = 0$, $[\varepsilon_{33}] = 0$ so that

$$[a_{11}] = (v_3 - G)\, [\gamma], \quad [a_2] = 0, \quad [a_3] = 0 \tag{3.2}$$

Let σ_{ij} be the physical components of the stress tensor in the coordinates ξ_i. We have $\sigma_{11} = \sigma_{22} = q$, $\sigma_{12} = \sigma_{23} = 0$, $\sigma_{13} = \tau$. The equations of motion are written in the form

$$\frac{\partial q}{\partial s_1} + q\, \frac{\partial \lambda_2}{\partial s_1} + \frac{\partial \tau}{\partial s_3} + \tau\, \frac{\partial}{\partial s_3} (2\lambda_1 + \lambda_2) - \sigma_2\, \frac{\partial \lambda_2}{\partial s_1} = \rho a_1. \quad \frac{\partial \sigma_2}{\partial s_2} + (\sigma_2 - q)\, \frac{\partial}{\partial s_2} (\lambda_1 + \lambda_3) = \rho a_2 \tag{3.3}$$

$$\frac{\partial \tau}{\partial s_1} + \tau\, \frac{\partial}{\partial s_1} (2\lambda_3 + \lambda_2) + \frac{\partial q}{\partial s_3} + q\, \frac{\partial \lambda_2}{\partial s_3} - \sigma\, \frac{\partial \lambda_2}{\partial s_3} = \rho a_3$$

Let $\mathbf{e}_i$ be the directions of the coordinate system ξ_i. According to the derivation formulas

$$\frac{\partial \mathbf{e}_1}{\partial s_1} = -\frac{\partial \lambda_1}{\partial s_2} \mathbf{e}_2 - \frac{\partial \lambda_1}{\partial s_3} \mathbf{e}_3, \quad \frac{\partial \mathbf{e}_2}{\partial s_2} = -\frac{\partial \lambda_2}{\partial s_3} \mathbf{e}_3 - \frac{\partial \lambda_2}{\partial s_1} \mathbf{e}_1$$

Since $\partial \mathbf{e}_1 / \partial s_1 = \partial \mathbf{e}_2 / \partial s_2 = 0$ on S, then $\partial \lambda_1 / \partial s_3 = \partial \lambda_2 / \partial s_3 = 0$.

From (3.3), the dynamical compatibility conditions on the maximum shear surface are

$$\left[\frac{\partial \tau}{\partial s_3}\right] - [\sigma_2]\, \frac{\partial \lambda_2}{\partial s_1} = \rho\, (v_3 - G)\, [\gamma], \quad \left[\frac{\partial \sigma_2}{\partial s_2}\right] + \left[(\sigma_2 - q)\, \frac{\partial}{\partial s_2} (\lambda_1 + \lambda_3)\right] = 0, \quad \left[\frac{\partial q}{\partial s_3}\right] + 2\tau \left[\frac{\partial \lambda_3}{\partial s_1}\right] - [\sigma_2]\, \frac{\partial \lambda_2}{\partial s_3} \tag{3.4}$$

As is shown above (Sect.2), $[\sigma_2] = 0$ in Mises theory. Moreover, $\partial\tau / \partial s_3 = \varkappa(\chi)\partial\chi / \partial s_3$ where $\varkappa = dk / d\chi$ is the plastic modulus. Equations (3.4) become

$$\varkappa\left[\frac{\partial\chi}{\partial s_3}\right] - \rho(v_3 - G)[\gamma] = 0, \quad \left[\frac{\partial\lambda_3}{\partial s_2}\right] = 0, \quad \left[\frac{\partial q}{\partial s_3}\right] + 2\tau\left[\frac{\partial\lambda_3}{\partial s_1}\right] = 0 \tag{3.5}$$

There results from (2.3) and the first equation in (3.5) that the velocity of shear wave propagation equals (V_n is the normal particle velocity on the shear surface)

$$G - V_n = \sqrt{\varkappa / \rho} \tag{3.6}$$

4. Tresca condition.

Let us first examine the case $\sigma_1 > \sigma_2 > \sigma_3$. The flow condition inthe coordinates ξ_i is written in the form $\tau = k$. According to the associated flow law $\varepsilon_{ij} = 0$, moreover, $\varepsilon_{13} = {}^1/_2\gamma$. There are therefore five kinematic equations that yield five relationships on the weak velocity discontinuity surface. Investigating them results in the deduction that the discontinuity in ε_{ij} is possible only on maximal shear surfaces $\xi_3 = \text{const}$ or $\xi_1 = \text{const}$.

Let us consider discontinuities on the surface $\xi_3 = \text{const}$. It follows from the first equations of (3.3) and (2.3) that if $[\sigma_2] = 0$, then $[\gamma] \neq 0$ under the condition (3.6). If $[\sigma_2] \neq 0$ because of the boundary conditions, then $[\gamma] \neq 0$ for $\rho(v_3 - G)^2 \neq \varkappa$.

Now, let $\sigma_1 = \sigma_2 > \sigma_3$ (the case $\sigma_1 > \sigma_2 = \sigma_3$ is considered analogously). In this case, the flow law imposes no new conditions on the weak discontinuity surface S as compared with Sect.1. Because of the condition $\sigma_1 = \sigma_2$ there is a principal direction of the tensor σ_{ij} tangent to an arbitrary surface at any point of this surface. Since the tensors σ_{ij} and ε_{ij} are coaxial, the selection of S is not constrained in any way.

Let us introduce the curvilinear coordinates α, β, ν, where we direct the ν axis along the normal to S, and the β axis along the principal direction tangent to S. The plasticity conditions are written in the form

$$f_1 = (\sigma_\nu - \sigma_\alpha)^2 + 4\tau_{\alpha\nu}^2 - 4k^2 = 0, f_2 = \sigma_\nu + \sigma_\alpha - 2\sigma_\beta - 2k = 0 \tag{4.1}$$

It is hence seen that $[\sigma_\alpha] = [\sigma_\beta] = 0$.

According to the associated flow law

$$\varepsilon_\nu = 2\mu_1(\sigma_\nu - \sigma_\alpha) + \mu_2, \quad \varepsilon_\alpha = 2\mu_1(\sigma_\alpha - \sigma_\nu) + \mu_2, \quad \varepsilon_\beta = -2\mu_2, \quad \varepsilon_{\alpha\nu} = 4\mu_1\tau_{\alpha\nu} \tag{4.2}$$

Since $[\dot{\varepsilon}_\beta] = 0$, then $[\mu_2] = 0$. Since $[\varepsilon_\alpha] = 0$, then $[\mu_1](\sigma_\alpha - \sigma_\nu) = 0$. Hence, discontinuities of ε_{ij} are possible only for $\sigma_\alpha = \sigma_\nu$. There results from (4.1) that $\tau_{\alpha\nu} = k$, i.e., S is a maximum shear surface.

Since $\sigma_1 = \sigma_2$, it follows $[\sigma_2] = 0$, and as is seen from (3.4) and (2.3) a discontinuity in γ is possible only under the condition (3.6).

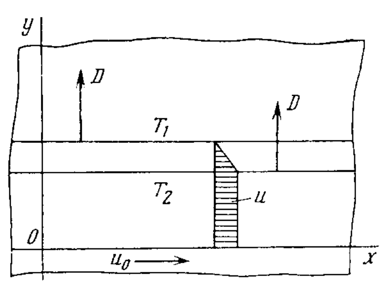

Fig.2

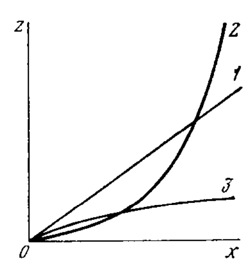

Fig.3

Let us note that in the case under consideration, as in the previous cases, the strain rate normal to S is continuous on S. It follows from (3.1) that the acceleration component normal to S is continuous on S.

The results obtained are evidently extended to convex and piecewise-linear plasticity conditions of general form.

5. Propagation of strain in rigidly plastic bodies.

The appearance of a different kind of discontinuity is possible on rigidly plastic boundaries (T). Let us first note that by applying the reasoning of Sect.1 to T, we arrive at the deduction that the surface T should be tangent to at least one principal direction of the tensor ε_{ij} at each point /4/. It can also be shown that all the constraints resulting from the flow law and obtained in Sect.2 and 4 /4/ are carried over to T. However, the question of the propagation velocity of rigidly plastic boundaries in the case when there is a jump in γ on them depends on whether the material is or is not in a neigborhood of T lying in the rigid zone, at the yield point.

For quasistatic flows it is customary to consider the material to be at the yield point /5/ in the rigid domain adjoining T.

If such a representation is taken, then all the deductions about the propagation velocity of a weak velocity discontinuity surface, that were obtained above, carry over to the rigidly plastic boundaries. However, it is evidently false to consider the material in the neigborhood of the rigidly plastic boundary in the rigid domain as being at the yield point in dynamic flows in all cases.

In conclusion, we consider plane shear wave propagation in an initially homogeneous half-plane $y \geqslant 0$ (Fig.2). Let the edge of the half-plane be clamped rigidly to a nondeformable edging, whose velocity is given. Let $u(t, y)$ denote a unique velocity projection different from zero on the $0x$ axis. We then have the boundary condition $u(t, 0) = 0$ for $t \leqslant 0, u(t, 0) = u_0(t)$, $u_0(0) = 0$ for $t \geqslant 0$. We moreover assume that $u(0, y) = \chi(0, y) = 0$.

The equations of motion and the hardening condition are written in the form

$$\frac{\partial \tau_{xy}}{\partial y} = \rho \frac{\partial u}{\partial t}, \quad \frac{\partial \chi}{\partial t} = \left| \frac{\partial u}{\partial y} \right| \tag{5.1}$$

Since $\tau_{xy} = -k(\chi)$, there should be $\partial u / \partial y < 0$.

Equations (5.1) take the form

$$\frac{\partial u}{\partial t} + \frac{\varkappa}{\rho} \frac{\partial \chi}{\partial y} = 0, \quad \frac{\partial u}{\partial y} + \frac{\partial \chi}{\partial t} = 0 \tag{5.2}$$

This is a hyperbolic system. The equations of the characteristics are

$$dy = \pm \sqrt{\varkappa / \rho}\, dt, \quad u \pm U(\chi) = \text{const}, \quad U(\chi) = \int_0^\chi \sqrt{\varkappa / \rho}\, d\chi$$

(the plus sign is for the first family, and the minus is for the second).

The rigid domain is separated from the deformable rectilinear first-family characteristic in the ty plane. Since $u = \chi = U = 0$ in the rigid domain then the integral $u = U(\chi)$ holds in the flow domain.

The first family characteristics are lines along which $u = \text{const}$ and $\chi = \text{const}$. We find from $u(t, y) = U(\chi)$ and (5.2)

$$\partial u / \partial y = -\sqrt{\rho / \varkappa} \cdot \partial u / \partial t \quad \text{and} \quad \partial u / \partial y \,|_{y=0} = \sqrt{\rho / \varkappa}\, u_0'(t)$$

It is hence seen that the strain is propagated from the boundary as long as the edging is accelerating. For $u_0' \leqslant 0$, a rigid zone starts to be propagated from the boundary into the half-plane depths, and the deformable domain in the form of a wave packet continues to be propagated into the depth of the half-plane (Fig.2).

Since the deformable domain is separated from the rigid domains by first-family characteristics, the rigidly plastic boundaries are propagated at the velocity $\sqrt{\varkappa / \rho}$. The appearance of a discontinuity $\gamma = \partial u / \partial y$ on them depends on the boundary condition $u_0(t)$. If $u'_0(0) \neq 0$, a jump γ occurs at the leading front of the wave T_1. Analogously, the jump in γ can also occur on the trailing front of the wave T_2.

Let us note that if the hardening curve is convex downward, $\varkappa'(\chi) = k''(\chi) > 0$, an envelope of the first-family characteristics exists in the upper half-plan ty, and therefore, the continuous solution constructed does not hold. In this case a shock, a line of strong velocity discontinuity, appears.

6. Structure of a strong velocity shock. It was shown in /4,6/ that a strong velocity discontinuity can be obtained for $\varkappa = \text{const}$ as the limit of a layer of finite thickness h when $\gamma \to \infty$ as $h \to 0$. This method is not suitable for $\varkappa \neq \text{const}$. In this connection, we examine the steady flow of a viscous rigidly platic medium in an infinite plane under pure shear.

Let the velocity projections on the axes x, y be $u = u(y)$, $v = \text{const}$. Then $\sigma_x = \sigma_y = \text{const}, \tau_{xy} = \tau(y)$. The equation of motion, the hardening condition, and the flow law are written in the form

$$\frac{d\tau}{dy} = \rho v \frac{du}{dy}, \quad v \frac{d\chi}{dy} = \frac{du}{dy}, \quad \tau = k(\chi) + s(u') \tag{6.1}$$

($s(u')$, the viscous stress, is a given function of u', $u' = du / dy$, and $s(0) = 0, s'(u') > 0$). We have the following boundary conditions: $u = u' = \varkappa = 0$ for $y \to -\infty$ and $u = u_1$, $u' = 0$, $\chi = \chi_1$ as $y \to \infty$.

We find

$$\tau = \rho v u + k_0, \quad u = v\chi \tag{6.2}$$

from the first two equations and the boundary condition as $y \to -\infty$. Substituting (6.2) in the last equation of (6.1), we obtain

$$s(u') = \rho v u - k(u / v) + k_0 = z(u) \tag{6.3}$$

Since $s(u') > 0$, it follows $z > 0$. Taking into account that $z(0) = 0$, we require that $dz/du > 0$ for $u = 0$. We obtain $\rho v^2 - \varkappa(0) > 0$, or $v > \sqrt{\varkappa(0)/\rho}$.

The boundary condition as $y \to \infty$ requires that $z(u_1) = 0$. This requirement results in an equation governing v, the wave propagation velocity

$$k(u_1/v) - k_0 = \rho v u_1$$

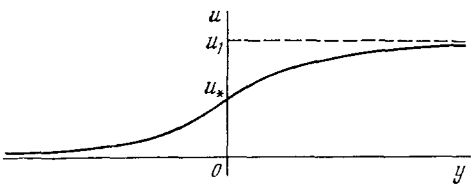

Fig.4

The mutual location of the lines $z = \rho v u$ and $z = k(u/v) - k_0$ is shown in Fig.3 (*1* is the line $z = \rho v^2\chi$, *2* is the line $k(\chi) - k_0$ for $k'' > 0$, *3* is the line $k(\chi) - k_0$ for $k'' < 0$). If the curve $k(\chi)$ is convex downward, then $z(u)$ vanishes only for $u = 0$. If $k(\chi)$ is convex upward, then $z(u)$ has still another zero at the point $u = u_1$.

Therefore, the problem has a solution only for $k''(\chi) > 0$ when $k(\chi)$ is convex downward.

Let us also note that in this latter case dz/du vanishes at the point $u = u_*$ determined from the equation $\varkappa(u/v) = \rho v^2$, where $u_* \in (0, u_0)$.

Let $f(z)$ be a function inverse to $s(u')$. Then we find a dependence of y on u from (6.3)

$$y = \int_{u_*}^{u} \frac{du}{f(\rho v u - k(u/v) + k_0)} \tag{6.4}$$

It is hence seen that $y \to -\infty$ as $u \to 0$ and $y \to +\infty$ as $u \to u_1$. The dependence $u(y)$ is shown in Fig.4.

It is interesting to note that the results of Sect.6 are independent of whether the material is or is not outside the shock at the yield point.

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